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A NEW SPECIAL FUNCTION AND FRACTIONAL CALCULUS

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ABSTRACT

In recent year's many special functions given by mathematicians, here a new function termed as M- function has been introduced. This Function is a particular case of H-function given in [2,3]. This function is important because hypergeometric function and Mittag-Leffler function follow as particular cases and these functions have great significance in the context of problems in physics, biology, engineering and applied sciences. It is to be noted that Mittag-Leffler [4,5] function occurs as solution of fractional integral equations in those subjects. In this paper we have also obtained the fractional integration and fractional differentiation of M- function.

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KEYWORDS: Fractional Calculus, New special function and Riemann-Liouville Operator..

INTRODUCTION

We give the new special function, called $\mathbf{M}-$ function, which is the most generalization of New Generalized Mittag-Leffler Function . Here, we give first the notation and the definition of the New Special $\mathbf{M}-$ function, introduced by the author as follows:

$$ {}^{\alpha,\beta,\gamma,\delta,\rho}_{\quad p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}(t) = \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots\ (a_p)_n\,(\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(ct)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} \qquad (1)$$

There are p upper parameters $a_1, a_2, \dots a_p$ and q lower parameters $b_1, b_2, \dots b_q, \alpha, \beta, \gamma, \delta, \rho \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\rho) > 0, Re(\alpha\gamma - \beta) > 0$and $(a_j)_k\ (b_j)_k$ are pochammer symbols and $k_1, \dots k_p$, $l_1, \dots l_q$are constants. The function (1) is defined when none of the denominator parameters $b_j s,\ j = 1,2, \dots q$ is a negative integer or zero. If any parameter a_j is negative then the function (1) terminates into a polynomial in (t).

RELATIONSHIP OF THE ${}^{\alpha,\beta,\gamma,\delta,\rho}_{\quad p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}$FUNCTION AND OTHER SPECIAL FUNCTIONS:

In this section, we define relationship of **M** function and various special functions.

(i). For $k_1 = a, k_2 \dots k_p = 1, l_1, \dots l_q = 1, \delta = 1$ and $\rho = 1, c = 1$, we defined relationship of **M** function and various special functions. .

The $\mathbf{M}-$ function reduces to New Generalized Mittag-Leffler Function [6]

$$ {}^{\alpha,\beta,\gamma,1,1}_{\quad 1}\mathbf{M}_1^{a,1}(t) = t^{\alpha\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n (t)^{\alpha n}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} = t^{\alpha\gamma-\beta-1} E_{\alpha,\alpha\gamma-\beta}^{\gamma}[at^\alpha] \qquad (2)$$

(ii). We take $\gamma = 1$, in (2) obtained Generalized Mittag-Leffler Function [10], we get

$$ {}^{\alpha,\beta,1,1,1}_{\quad 1}\mathbf{M}_1^{a,1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{(n+1)\alpha-\beta-1}}{\Gamma\big((n+1)\alpha-\beta\big)} = t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}[at^\alpha] \quad (3)$$

(iii). Further $\beta = \alpha - 1$,, in(3), this **M** function converts Mittag-Leffler Function [6,7], we have

$$ {}_{1}^{\alpha,\alpha-1,1,1,1}\mathbf{M}_{1}^{a,1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n\alpha}}{\Gamma(n\alpha+1)} = E_\alpha[at^\alpha] \qquad (4) $$

(iv). When $a = 1$ and $\beta = \alpha - \beta$ in (4) then the **M** function treats as Agarwal's Function [1]

$$ {}_{1}^{\alpha,\alpha-\beta,1,1,1}\mathcal{M}_{1}^{1,1}(t) = \sum_{n=0}^{\infty} \frac{(t)^{n\alpha+\beta-1}}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}[t^\alpha] \qquad (5) $$

(ix). On substituting $\alpha = 1, \beta = -\beta$ in (3), we get Miller and Ross Function [5].

$$ {}_{1}^{1,-\beta,1,1,1}\mathcal{M}_{1}^{a,1}(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n+\beta}}{\Gamma(n+\beta+1)} = E_t[\beta, a] \qquad (6) $$

MATHEMATICAL PREREQSITIES

The Riemann-Liouville fractional integral of order $v \in C$ is defined by Miller and Ross (1993, p.45;)

$$ {}_0D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u)du, \qquad \textbf{(3.1)} $$

where Re(v)>0. Following Samko et al. (1993, p. 37) we define the fractional derivative for $\alpha > 0$ in the form

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)du}{(t-u)^{\alpha-n+1}}, \qquad (n = [\mathrm{Re}(\alpha)]+1), \qquad \textbf{(3.2)} $$

Where $[\mathrm{Re}(\alpha)]$ means the integral part of $\mathrm{Re}(\alpha)$.

FRACTIONAL INTEGRAL AND FRACTIONAL DIFFERENTIAL OF THE M-FUNCTION

Let us consider the fractional Riemann − Liouville (R-L) integral operator, as in [5,8] (for lower limit $a = 0$ with respect to variable z) of the **M** -function (1).

$$ {}_0D_t^{-v}\, {}_{p}^{\alpha,\beta,\gamma,\delta,\rho}\mathbf{M}_{q}^{k_1,\dots k_p,l_1,\dots l_q;c}(t) $$

$$ = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} \sum_{n=o}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(ct)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)}\, dt $$

$$ = \frac{1}{\Gamma(v)} \sum_{n=o}^{\infty} \frac{(a_1)_n \dots (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} \times $$

$$ \int_0^z (z-t)^{v-1}\, (t)^{(n+\gamma)\alpha-\beta-1}\, dt $$

$$= \frac{1}{\Gamma(v)} \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n \ (c)^{(n+\gamma)\alpha-\beta-1}}{l_1^n \dots l_q^n \ n! \, \Gamma((n+\gamma)\alpha-\beta)} z^{(n+\gamma)\alpha-\beta+v-1} \times$$

$$B((n+\gamma)\alpha-\beta, v)$$

$$= \frac{1}{\Gamma(v)} \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n \ (c)^{(n+\gamma)\alpha-\beta-1}}{l_1^n \dots l_q^n \ n! \, \Gamma((n+\gamma)\alpha-\beta)} z^{(n+\gamma)\alpha-\beta+v-1} \times$$

$$\frac{\Gamma((n+\gamma)\alpha-\beta)\Gamma(v)}{\Gamma((n+\gamma)\alpha-\beta+v)}$$

$$= \frac{\Gamma((n+\gamma)\alpha-\beta-n)}{\Gamma((n+\gamma)\alpha-\beta+v-n)} z^v \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n \ (c)^{(n+\gamma)\alpha-\beta-1}}{l_1^n \dots l_q^n \ n! \, \Gamma((n+\gamma)\alpha-\beta)}$$

$$\times z^{(n+\gamma)\alpha-\beta-1} \frac{((n+\gamma)\alpha-\beta-n)_n}{((n+\gamma)\alpha-\beta+v-n)_n}$$

$$= \frac{\Gamma((n+\gamma)\alpha-\beta-n)}{\Gamma((n+\gamma)\alpha-\beta+v-n)} z^v \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n \ (c)^{(n+\gamma)\alpha-\beta-1}}{l_1^n \dots l_q^n \ n! \, \Gamma((n+\gamma)\alpha-\beta)}$$

$$z^{(n+\gamma)\alpha-\beta-1} \quad \frac{((n+\gamma)\alpha-\beta-n)_n}{((n+\gamma)\alpha-\beta+v-n)_n}$$

$${}_0D_t^{-v} \, {}^{\alpha,\beta,\gamma,\delta,\rho}_{\ p}\mathbf{M}_q^{k_1,\dots k_p, l_1,\dots l_q; c}(t) = \frac{\Gamma((n+\gamma)\alpha-\beta-n)}{\Gamma((n+\gamma)\alpha-\beta+v-n)} \ z^v \, {}^{\alpha,\beta,\gamma,\delta,\rho}_{\ p}\mathbf{M}_q^{k_1,\dots k_p, l_1,\dots l_q; c}$$

$$(a_1 \dots a_p, \gamma, \delta, \left(((n+\gamma)\alpha-\beta-n) \ \right);, b_1 \ \dots \ b_{q,} \rho,, ((n+\gamma)\alpha-\beta+v-n); \ z) \quad (4.1)$$

Riemann – Liouville Fractional derivative of **M** -Function which indices p, q are increased to $(p+1), (q+1)$. Analogously, Riemann – Liouville fractional derivative operator [5,8] of the **M** -Function with respect to z.

$$D_z^v \, {}^{\alpha,\beta,\gamma,\delta,\rho}_{\ p}\boldsymbol{M}_q^{k_1,\dots k_p, l_1,\dots l_q; c}(t) = \frac{1}{\Gamma(n-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} \, {}^{\alpha,\beta,\gamma,\delta,\rho}_{\ p}\boldsymbol{M}_q^{k_1,\dots k_p, l_1,\dots l_q; c}(t) \, dt$$

$$D_z^v \, {}^{\alpha,\beta,\gamma,\delta,\rho}_{\ p}\boldsymbol{M}_q^{k_1,\dots k_p, l_1,\dots l_q; c}(t) = \frac{1}{\Gamma(n-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1}$$

$$\sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(ct)^{(n+\gamma)\alpha-\beta-1}}{n! \, \Gamma((n+\gamma)\alpha-\beta)} dt$$

$$= \frac{1}{\Gamma(n-v)} \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots \ (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{1}{n! \, \Gamma((n+\gamma)\alpha-\beta)} \ c^{(n+\gamma)\alpha-\beta-1} \left(\frac{d}{dz}\right)^n$$

$$\int_0^z (z-t)^{n-v-1}\; t^{(n+\gamma)\alpha-\beta-1}\, dt$$

$$= \frac{1}{\Gamma(n-v)} \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots\ (a_p)_n\, (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} n!\, \Gamma\big((n+\gamma)\alpha-\beta\big)\, c^{(n+\gamma)\alpha-\beta-1} \left(\frac{d}{dz}\right)^n$$

$$z^{(n+\gamma)\alpha-\beta+n-v-1}\, B(n-v, (n+\gamma)\alpha-\beta$$

We use the modified Beta-function:

$$\int_a^b (b-t)^{\beta-1}\; (t-a)^{\alpha-1}\, dt \;= (b-a)^{\alpha+\beta-1}\, B(\alpha,\beta),\ \text{ for } R(\alpha)>0, R(\beta)>0$$

$$= \frac{1}{\Gamma(n-v)} \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots\ (a_p)_n\, (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} n!\, \Gamma\big((n+\gamma)\alpha-\beta\big)\, c^{(n+\gamma)\alpha-\beta-1} \left(\frac{d}{dz}\right)^n z^{(n+\gamma)\alpha-\beta+n-v-1}$$

$$\frac{\Gamma(n-v)\Gamma((n+\gamma)\alpha-\beta)}{\Gamma((n+\gamma)\alpha-\beta+n-v)} \quad (4.2)$$

Where $k+1>0, n-v>0$

Differentiation n times the term $z^{(n+\gamma)\alpha-\beta+n-v-1}$ and using again

$\Gamma(a+k)=(a)_k\Gamma(a)$, representation (4.2) reduces to

$$= \frac{1}{\Gamma(\alpha(n+\gamma)-\beta+n)} z^{-v} \sum_{n=o}^{\infty} \frac{(a_1)_n \ \dots\ (a_p)_n\, (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{1}{n!}$$

$$\times\ (cz)^{(n+\gamma)\alpha-\beta-1} (\alpha(n+\gamma)-\beta)_n \frac{}{\big((n+\gamma)\alpha-\beta-v\big)_n}$$

$$D_z^v\, {}^{\alpha,\beta,\gamma,\delta,\rho}{}_p\mathbf{M}_q^{k_1,\dots k_p, l_1,\dots l_q; c}$$

$\frac{1}{\Gamma(\alpha(n+\gamma)-\beta+n)}\, z^{-v}\, {}^{\alpha,\beta,\gamma,\delta,\rho}{}_p\mathbf{M}_q^{k_1,\dots k_p, l_1,\dots l_q; c}\, (a_1 \dots a_p, \gamma, \delta, \big(((n+\gamma)\alpha-\beta)\ \big);, b_1\ \dots\ b_{q,}\, \rho,, ((n+\gamma)\alpha-\beta-v);\ z)$ (4.3)

(4.3) gives a Riemann − Liouville fractional derivative of $\mathbf{M}-\mathbf{function}$, which indices p, q are increased to (p+1),(q+1).

CONCLUSION

In this present work, we have introduced a new special function and two new results in which we have obtained fractional Integration and Fractional Differentiation of $\mathbf{M}-\mathbf{function}$.

REFERENCES

[1] Carlson, B.C.: Special function of Applied Mathematics, Academic Press, (1977), New York

[2] Mathai, A.M.: A Handbook of Generalized Special Functions for Statistical and Physical Sciences, Clarendon Press, (1993), Oxford.

[3] Mathai, A.M and Saxena, R. K.: The H-function with Applications in Statistics and other Disciplines. John Wiley and Sons. Inc., (1978), New York.

[4] Mittag-Leffler , G.M. : Sur la nouvelle fonction $E_{\alpha}(x)$, C.R. Acad. Sci., Paris (Ser.II) 137 (1903), 554-558.

[5] Mittag-Leffler , G.M.: Sur la representation analytique d'une branche uniforme d'une fonction monogene, Acta Math. 29 (1905), 101-181.

[6] Miller K.S and Ross B.: An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, 1993.

[7] Oldhman K.B. and Spanier J.: The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974.

[8] Prabhakar, T.R. : A singular integral equation with generalized Mittag- Leffler function in the kernel, Yokohama Mathematical Journal 19 (1971), 7-15.

[9] Sharma, M.: Fractional integration and fractional differentiation of the M-series, Fract. Calc. Appl. Anal. 11 No.2 (2008), 187-192

[10] Saxena, R.K. , Mathai A.M. and Haubold, H.J. On generalized fractional kinetic equations, Physica A 344 (2004), 653-664.